Efficient Power Spectrum estimation using prewhitening and post coloring technique

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***Abstract*—The power spectrum estimation for a multichannel autoregressive process using prewhitened and postcoloring technique, which was originally developed for a single channel, is proposed. In order to make the extension, the Cholesky decomposition of the inverse autocorrelation matrix for a multichannel autoregressive process is discussed and the autoregressive model order selection for a multichannel process based on the exponentially em-bedded families criterion is introduced. The asymptotic mean and variance of the proposed estimator are derived. Compared to a filter-based autoregressive prewhiteneds multichannel power spectral estimator, the new estimator has less bias, i.e. higher resolution, and less overall mean square error for short data records due to the amelioration of end effects by the matrix prewhitener. It can serve as an excellent multichannel spectral estimator for processes exhibiting a wide dynamic range. Simulation results are given which show the advantage of the new estimator over a variety of common multichannel power spectral density estimators.**

***Index Terms*—AR matrix prewhitened and postcoloring power spectral density, multichannel AR process, dynamic range power spectral density estimation, consistent estimate.**

I. INTRODUCTION

Multichannel power spectral density (MPSD) estimation techniques are widely used in many applications, such as sonar, radar, geophysics and biomedicine. Similar to single channel power spectral density (PSD) estimation, there are basically two broad categories of MPSD estimators. One is the nonparametric approach, among which the Fourier-based estimators are the most popular [1–3]. The other is the parametric method, which assumes a model for the data. Spectral estimation then becomes a problem of estimating the parameters in the assumed model. The most commonly used model is the autoregressive (AR) model because accurate estimates of the AR parameters can be found by solving a set of linear equations [1,3,5]. Similar to the single channel case for short data records the Fourier-based methods can suffer from significant bias problems while AR model-based methods can suffer from inaccuracies in the model as well as from imprecise model order selection. Furthermore, some effective AR model-based approaches cannot be easily extended to the multichannel case [1,5]. In addition, as pointed out by Jenkins and Watts [2], spurious cross-correlation or spurious cross-spectral content may arise unless a prewhitening filter is applied before MPSD estimation.One such prewhitening filter was suggested by Thomson [6] for single channel PSD estimation.

The filter system function is given by $A(z) = 1 + \sum_{k=1}^{p} a(k) z^{-k}$ and the filter parameters a[1],a[2],a[3],…a[p]can be estimated from the data using any AR-model based method. Denoting the output of this FIR filter by u[n], a Fourier-based estimator is then used to generate the PSD estimate $\hat{P}_u(f)$. Finally, the PSD estimate of the original data is found as [6]

$$\hat{P}_x(f) = \frac{\hat{P}_u(f)}{|1 + \sum_{k=1}^{p} \hat{a}[k] \exp(-j2\pi fk)|^2} \qquad (1)$$

where $\hat{a}[1], \hat{a}[2], \ldots, \hat{a}[p]$ are the estimated AR filter parameters. We term this the AR prewhitened (ARPW) spectral estimator.

Because of the inconsistency of the definitions in the literature concerning MPSD estimation, the following definitions will be made. A complex multichannel sequence x[n] is defined as the complex L × 1 vectorx[n] = $[x_1[n], x_2[n], \cdots, x_L[n]]^T$

where $x_i[n]$ represents the data observed at the output of the ith channel and L is the number of channels. For a wide sense stationary (WSS) multichannel random process, the autocorrelation function (ACF) at lag k is defined as the L × L matrix function

$$R_X[k] = E[x[n+k]x^H[n]]$$

$$= \begin{bmatrix} r_{11}[k] & r_{12}[k] & . & .. & r_{1L}[k] \\ r_{21}[k] & r_{22}[k] & . & & r_{2L}[k] \\ : & : & : & & : \\ r_{L1}[k] & r_{L2}[k] & . & & r_{LL}[k] \end{bmatrix} \qquad (2)$$

where E[·] is the mathematical expectation, the superscript H denotes conjugate transpose and $r_{ij}[k]$ is the cross-correlation function (CCF) between $x_i[n]$ and $x_j[n]$ at lag k

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$$r_{ij}[k] = E[x_i[n+k]x^*_j[n]] \quad (3)$$

For multichannel data of N samples, the sample vector, which is NL × 1, is defined as

$$x = [x^T[0], x^T[1], \cdots, x^T[N-1]]^T \quad (4)$$

The NL × NL multichannel autocorrelation matrix of order N is defined as $Rx = E[xx^H]$ (5)

$$\begin{bmatrix} Rx[0] & Rx[-1] & \ldots & Rx[-N+1] \\ Rx[1] & Rx[0] & \ldots & Rx[-N+2] \\ : & & & \\ Rx[N-1] & Rx[N-2] & \ldots & Rx[0] \end{bmatrix}$$

From definition (2) it is seen that $R^H_x[k] = R_x[-k]$, so R_x is hermitian. Because the multichannel process is assumed to be wide sense stationary, Rx is also block Toeplitz. The power spectral density matrix or cross-spectral matrix is defined as

$$Px(f) = \begin{bmatrix} P_{11}(f) & P_{12}(f) & \ldots & P_{1L}(f) \\ P_{21}(f) & P_{22}(f) & \ldots & P_{2L}(f) \\ : & : & & : \\ P_{L1}(f) & P_{L2}(f) & \ldots & P_{LL} f) \end{bmatrix}$$

The diagonal elements Pii(f) are the PSDs of the individual channels or auto-PSDs, while the off-diagonal elements Pil(f) for i≠ l are the cross-PSDs between $x_i[n]$ and $x_l[n]$, which are defined as

$$Pil(f) = \sum_{k=-\infty}^{\infty} r_{il}[k] \exp(-j2\pi fk). \quad (6)$$

The magnitude squared coherence (MSC) between channels i and j is a quantity that indicates whether the spectral amplitude of the process at a particular frequency in channel i is associated with large or small spectral amplitude at the same frequency in channel j. It is defined

$$|\gamma_{ij}(f)|^2 = \frac{|P_{ij}(f)|^2}{P_{ii}(f)P_{jj}(f)} \quad (7)$$

A classic Fourier-based spectral estimator is the periodogram, which is given as the L × L matrix

$$P_{PER}(f) = \frac{1}{N} X(f)X^H(f) \quad (8)$$

where the Fourier transform is the L × 1 vector

$$X(f) = \sum_{n=0}^{N-1} x[n]\exp(-j2\pi fn). \quad (9)$$

The multichannel pth order AR model is defined as

$$x[n] = \sum_{i=1}^{p} A[i]x[n-i] + u[n] \quad (11)$$

where A[1],A[2], · · · ,A[p] are L × L AR coefficient matrices and u[n] is the excitation white noise or $R_u[k] = \Sigma\delta[k]$

and Σ is the L × L excitation noise covariance matrix with δ [·] being the discrete delta function.

The ARPW estimator given in (1) is readily extended to the multichannel case. With the notations defined above, the multichannel version of (1) is

$$\hat{P}_x(f) = \hat{A}^{-1}(f)\hat{P}_u(f)\hat{A}^{-H}(f) \quad (11)$$

are the estimated multichannel AR filter parameters. In addition to reducing spurious cross-spectral content, this prewhitened spectral estimator also gives an auto-PSD spectral estimate with much less bias than a Fourier-based spectral estimator. This is because the prewhitener reduces the dynamic range of the PSD. However, this estimator is still inferior to the method proposed in this paper. Instead of the FIR prewhitening filter, the proposed estimator uses a prewhitening matrix, which is essentially a time-varying filter that is less susceptible to end effects. The new estimator for a single channel PSD has been proposed in [4], while in this paper it is extended to MPSD estimation. Assume the multichannel

$$x[n] = Ac\exp(j2\pi f_0 n) + w[n],\ n = 0, 1, \ldots, N-1 \quad (12)$$

where x[n] is an L×1 vector, Ac is an L×1 complex amplitude to be estimated, f0 is a known frequency,and $w = [w^T[0], w^T[1], \ldots, w^T[N-1]]^T$ is a NL × 1 complex Gaussian noise vector with zero mean and known NL × NL covariance matrix Rw . The ML estimate of Ac is (13)

$$\hat{A}_c = (E_0^H R_w^{-1} E_0)^{-1} E_0^H R_w^{-1} X$$

where x is the data sample vector given by (4) and $E0 = [I_L, I_L \exp(j2\pi f0), \cdots, I_L \exp(j2\pi f0 (N-1)]^T$ (NL × L) with IL being an L × L identity matrix. The L × L covariance matrix of this estimator is [1]

$$C_{\hat{A}c} = (E_0^H R_w^{-1} E_0)^{-1} \quad (14)$$

Therefore, for a general WSS multichannel random process x[n] the MVSE is defined as

$$PMV(f) = p(E^H(f)\, \hat{R}_x^{-1} E(f))^{-1} \quad (15)$$

where R^x is the estimated pL × pL ACM of x and $E(f) = [I_L, I_L \exp(j2\pi f), \cdots, I_L \exp(j2\pi f(p-1))]^T$ (pL × L).

2. Multichannel AR Matrix Prewhitened SpectralEstimate

2.1 Motivation for the New Estimator and proposed system:

Similar to the derivation of the MVSE discussed in the previous section, the data structure of (12) is used. However, we now assume that Ac is an L × 1 complex Gaussian random vector with zero mean and unknown covariance matrix CA, which is to be estimated. We also assume that Ac is independent of the noise w[n] .We note that the basic data assumption here is that the noise samples w[0],w[1], · · · ,w[N −1] are correlated and hence a simple ML result for independent and identically distributed samples does not apply. As derived in Appendix A, the ML estimator of C_A is $\hat{C}_A = S_1 - S_2$ (16)

$$S1 = (E_o^H R_w^{-1} E_o)^{-1} E_o^H R_w^{-1} XX^H R_w^{-1} E_o (E_o^H R_w^{-1} E_o)^{-1}$$

Since Ac is independent of the noise vector w, the ACM of x is

$$E(XX^H) = E((E_o A_c + w)(A_c^H E_o^H + W^H))$$

$$= E_o C_A E_o^H + R_w$$

As a result, the expectation of S1 is

$$E((E_o^H R_w^{-1} E_o)^{-1} E_o^H R_w^{-1} XX^H R_w^{-1} E_o (E_o^H R_w^{-1} E_o)^{-1}) \quad (17)$$

Since from(14), $(E_o^H R_w^{-1} E_o)^{-1}$ is the covariance matrix of the noise at the narrowband filter output, $C_A + (E_o^H R_w^{-1} E_o)^{-1}$ will be the sinusoid plus noise covariance matrix at the filter output. Furthermore, S2 can be thought of as an approximate estimate of the noise covariance matrix at the filter output. So S1−S2 will be an estimate of the sinusoid covariance matrix or CA. Since the expected value of S1 is the overall power at the output of a narrowband filter, it can be used to indicate spectral content on average. Suppose the data does not consist solely of a sinusoid in noise, but more generally a wide sense stationary random process. Then we can define a new spectral estimator based on S1 as

$$\hat{P}_X(f) = N(E^H(f)\hat{R}_X^{-1}E(f))^{-1} E^H(f)\hat{R}_X^{-1} XX^H \hat{R}_X^{-1} E(f)(E^H(f)\hat{R}_X^{-1}E(f))^{-1} \quad (18)$$

for a single channel, it can be simplified to

$$\hat{P}_x(f) = N\left|\frac{e^H(f)\hat{R}_x^{-1}X}{e^H(f)\hat{R}_x^{-1}e(f)}\right|^2 \quad (19)$$

where $\hat{R}_x$ is an estimate of the N × N ACM of the single channel process and e(f) =[1, exp (j2πf),…, exp (j2πf (N − 1))]^T (N × 1). This is the ARMPW estimator proposed in [4]. Although the new multichannel spectral estimator is for complex data, it is also applicable to real data.

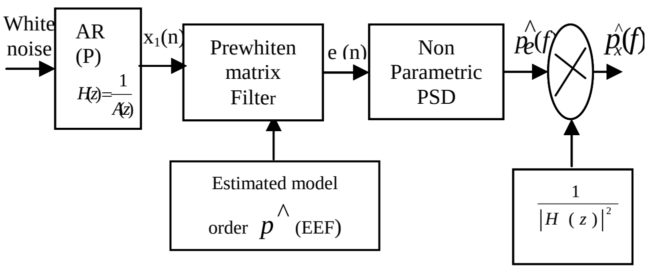

Figure 1. Proposed system for MARMPW spectral estimator

2.2 Properties of the MARMPW spectral estimator

The MARMPW spectral estimator is asymptotically unbiased. This means that as N → ∞ , we have E[P^x (f)] → Px (f). To see this, first note that if the autocorrelation matrix is a continuous function of a fixed number of parameters, then substituting the consistent estimates of these parameters, we have R^x → Rx in probability. If Rx is invertible and hence is a continuous function of its elements, then $\hat{R}^{-1}_x \to R^{-1}_x$ in probability [7]. Finally, by Slutsky's theorem [7] we can replace R^x by Rx in (18) to yield

$$\hat{P}_X(f) = N(E^H(f)R_X^{-1}E(f))^{-1} E^H(f)R_X^{-1}XX^H R_X^{-1}E(f)(E^H(f)R_X^{-1}E(f))^{-1}. \quad (20)$$

Taking the expectation we have

$$E[\hat{P}_X(f)] = N(E^H(f)R_X^{-1}E(f))^{-1} \quad (21)$$

$$= \left(\frac{E^H(f)}{\sqrt{N}} R_X^{-1} \frac{E(f)}{\sqrt{N}}\right)^{-1}$$

Since Rx is a hermitian and block Toeplitz matrix, it is well known that as N → ∞ [9,10]

$$R_X^{-1} \to \sum_{i=0}^{N-1} \frac{E(f_i)}{\sqrt{N}} P_X^{-1}(f_i) \frac{E^H(f_i)}{\sqrt{N}} \quad (22)$$

where fi = i/N. Therefore we have for fk = k/N that as N →∞

$$\frac{E^H(f_k)}{\sqrt{N}} R_X^{-1} \frac{E(f_k)}{\sqrt{N}} \to \frac{E^H(f_k)}{\sqrt{N}} \sum_{i=0}^{N-1} \frac{E(f_i)}{\sqrt{N}} P_X^{-1}(f_i) \frac{E(f_k)}{\sqrt{N}}$$

$$= P_X^{-1}(f) \quad (23)$$

This follows from $E^H(f_i)E(f_k) = NI_L\delta_{ik}$, where δ_{ik} is the Kronecker delta. Thus, we have finally from (21) and (23) that for $f = f_k = k/N$

$$E[\hat{P}_X(f)] \to P_X(f_k) \quad \text{then as } N \to \infty$$

This by a continuity argument, we have $E[\hat{P}_X(f)] \to$ Px (f) for continuous spectra. Thus, the MARMPW spectral estimator is an asymptotically unbiased estimator. By a similar argument (see Appendix B) it can be shown that as N → ∞, var [$\hat{P}_{ij}$ (f)] → P_{ii} (f) P_{jj} (f). This is the same asymptotic variance as for the multichannel periodogram. 3 *Implementation of the ARMPW*

3.1 Estimation of the Matrix Prewhitener:

As explained in [4] the inverse of he ACM R^{-1}_x in (18) acts as the prewhitener. In order to estimate this autocorrelation matrix prewhitener the Cholesky decomposition of R^{-1}_x for a complex WSS random process is derived in Appendix C. It is given as

$$R_X^{-1} = \Phi^H P^{-1} \Phi \quad (24)$$

where Φ is the NL × NL lower triangular matrix

$$\Phi = \begin{bmatrix} I & 0 & 0 & \cdots 0 \\ A_1[1] & I & 0 & \cdots 0 \\ : & & & \\ : & & & \\ A_{N-1}[N-1] & A_{N-1}[N-2] & & \cdots & I \end{bmatrix} \quad (25)$$ ix

$$P = \mathrm{diag}(\Sigma_0, \Sigma_1, \ldots, \Sigma_{N-1}). \tag{26}$$

Here I is the L × L identity matrix, $A_j[i]$ is the ith filter parameter for an AR model of order j, and Σ_j is the corresponding excitation noise covariance matrix. This decomposition is valid for any process. However, if the process is an AR process of order p, then for j ≥ p we have

$$A_j[i] = A_p[i] \quad i = 1, 2, \cdots, p$$
$$= 0 \quad i > p$$
$$\Sigma_j = \Sigma_p.$$

In order to obtain a statistically stable prewhitening matrix we restrict the order of the maximum AR model to p, where p << N.

3.2 AR model order selection for R_x

We consider first real AR models. The extension to complex is then given. For the selection of the AR model order, even though an arbitrary WSS random process is representable as an AR process of order p = ∞, the usual model order selection rules appear to choose reasonable finite order approximations. The commonly used methods are Akaike's information criterion (AIC) [11] and the minimum description length (MDL) criterion [12]. In this paper we propose to use a method recently described in [8], which is based on exponentially embedded families (EEF). It has been shown to have better performance than AIC and the MDL for AR model order estimation. For real data, it proceeds as follows [8].

$$\hat{P} = \arg \max_{1 \le i \le p_{max}} \{(L_{Gi}(X) - i[\ln(\frac{L_{Gi}(X)}{i}) + 1])u(\frac{L_{Gi}(X)}{i} \tag{28}$$

$$L_{Gi}(X) = (N-i)\ln\left(\frac{\det(\hat{\Sigma}_o)}{\det(\hat{\Sigma}_i)}\right) \tag{29}$$

n Where $\hat{\Sigma}_o$ and $\hat{\Sigma}_i$ are L×L conditional ML excitation noise covariance matrix estimators of the AR model of order i and the reference model respectively, given as [3]

$$\hat{\Sigma}_i = \frac{1}{N-i} X_i^T (I - H_i (H_i^T H_i)^{-1} H_i^T) X_i \tag{30}$$

$$\hat{\Sigma}_0 = \frac{1}{N-i} X_i^T X_i \tag{31}$$

Here I is the (N − i) × (N − i) identity matrix.

4 Summary of the New Estimator algorithm

The MARMPW spectral estimator is now summarized.

1. Choose a maximum AR model order p max.

2. using the EEF model order estimator of (28) obtain p̂.

3. Estimate the AR model parameters for the different order AR models j = 0, 1, · · · , p̂ as, $\{\hat{\Sigma}_O\}, \{A\hat{}[1], \hat{\Sigma}_1\}, \{\hat{A}_{\hat{P}}[1], \hat{A}_{\hat{P}}[2], \hat{A}_{\hat{P}}[3], \ldots, \hat{A}_{\hat{P}}[\hat{P}], \hat{\Sigma}[\hat{P}]\}$ (27)

In the simulations to follow we have used the autocorrelation or Yule-Walker method and so the AR model parameters for all the lower order models are available.

4. Segment the data into K equal length blocks, with K≥ L

5. Construct the N′L × N′L matrix $\hat{R}_x^{-1}$ using the estimated parameters.

6. Compute the MARMPW spectral estimate of (18) for each block and average them to get the final estimate.

5 Computer Simulation Results

In this section we present some computer simulations intended to show the bias, i.e. resolution, and mean square error (MSE) advantages of the proposed spectral estimator.. The EEF method is used to determine the AR model order.

We compare a two-channel process with both channels being two closely spaced complex sinusoids. The signal of channel 1 is

where u(x) is the unit step function and LGi(x) is the generalized

$$x1[n] = \exp(j(2\pi f_1 n + \varphi_1)) + \exp(j(2\pi f_2 n + \varphi_2)) + w_1[n],$$

n =0,1,2…,N-1.

and the signal of channel 2 is

$$x_2[n] = \exp(j(2\pi f_1 n + \varphi_1 + \pi/3)) + \exp(j(2\pi f_2 n + \varphi_2 + \pi/3)) + w_2[n],$$

n =0,1,2…,N-1.

where f1 = 0.22, f2 = 0.28, φ_1 and φ_2 are independent uniformly distributed random phases, w1[n] and w2[n] are two independent complex white Gaussian noise (WGN) processes with variance of 0.01. A data record of N = 256 points was segmented into 4 blocks with each block being N′ = 64, which is insufficient for Fourier-based MPSD estimators to resolve the two sinusoids. For this short data record case EEF is unable to give correct AR order which should be 2, since the data is approximately only two complex sinusoids.

6 Conclusions

As expected, the non parametric estimate does not resolve the two peaks in the true spectrum and suffers from leakage at high frequencies. However, the combined non parametric and parametric estimate resolves two peaks with cease and follows the true spectrum quite well. Therefore, the use of parametric methods as pre processor is highly recommended especially in wide dynamic range power spectrum estimation.

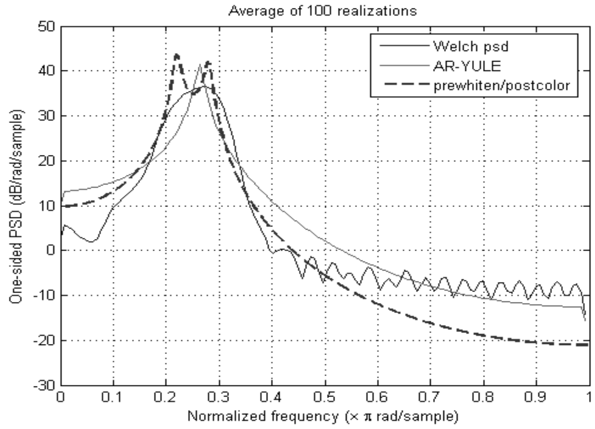

Figure 2. Comparison of PSD estimate of Prewhiten/Postcolor (MARMPW) to Welch and AR-YULE for two sinusoids in WGN with frequencies f_1=0.22 and f_2= 0.28.

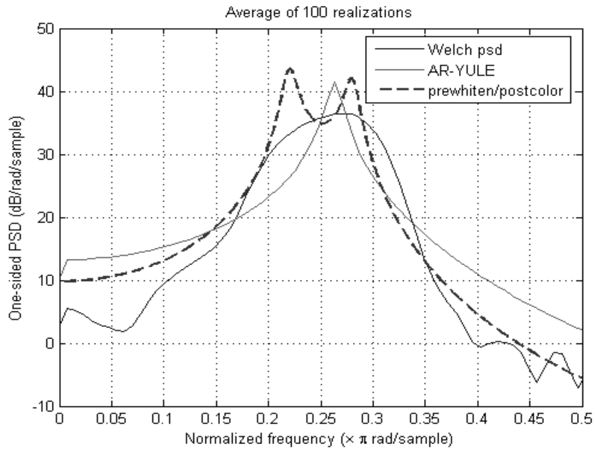

Figure 3. Expanded Version of Figure 2

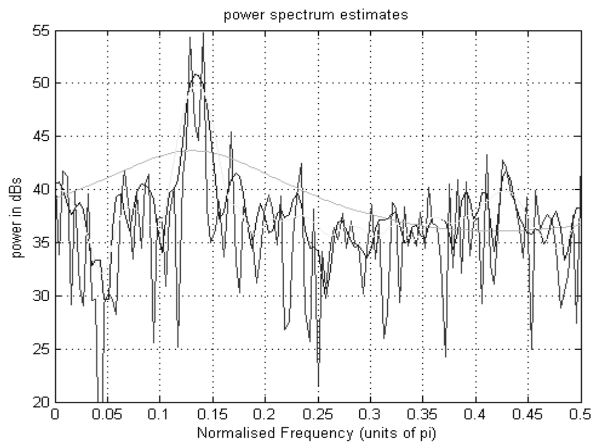

Figure4: Comparison of PSD estimates of Prewhiten/Postcolor (MARMPW) to Welch and AR-YULE for two sinusoids in WGN with frequencies f_1=0.13 and f_2= 0.14

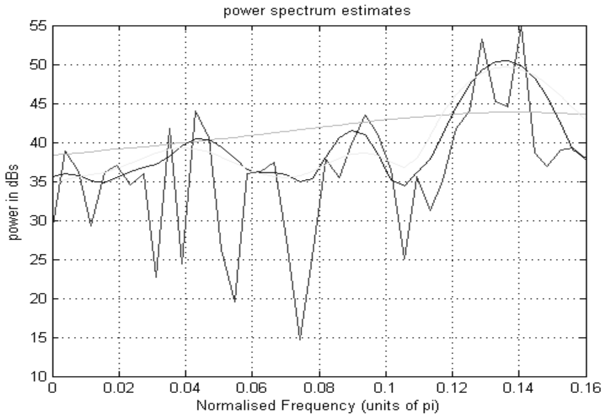

Figure5: Expanded version of Figure4

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REFERENCES

[1] S. M. Kay, Modern Spectral Estimation Theory and Application. Englewood Cliffs, NJ: Prentice-Hall, 1988.

[2] Gwilym M. Jenkins and Donald G. Watts, Spectral Analysis and its Applications. Holden-Day, 1968.

[3] Gregory C. Reinsel, Elements of Multivariate Time Series Analysis. Springer-Verlag,1993.

[4] S. M. Kay and Cuichun Xu, "A wide dynamic range Spectral estimator," Submitted to IEEE Trans. Signal Processing, 2005.

[5] Taiknang ning and C. L. Nikias, "Multichannel AR spectrum estimation: the optimum approach in the reflection coefficient domain," IEEE Trans. Acoust. Speech, Signal Processing, vol. ASSP-34, no. 5, pp 1139–1152, Oct. 1986.

[6] Thomson, D.J., "Spectrum estimation and harmonic analysis," Proceedings of the IEEE, Vol. 70, pp. 1055–1096, 1982.

[7] R.J. Serfling, Approximation Theorems of Mathematical Statistics. J.Wiley, 1980, New York.

[8] S. M. Kay, "Exponentially embedded families: A new approach to model order estimation", IEEE Trans. Aero and Elect., vol. 41, pp. 333–345, Jan. 2005.

[9] S. M. Kay, Fundamentals of Statistical Signal Processing: Detection Theory. Prentice-Hall, Englewood Cliffs, NJ, 1998.

[10] W.A.Fuller, Introduction to Statistical Time Series,J. Wiley, 1996, New York

[11] H. Akaike, "A new look at the Statistical Model Identification," IEEE Trans. Automat.Contr.,vol. AC-19, pp. 716-723, 1974.

[12] J. Rissanen, "Modeling by shortest data description," Automatica, vol. 14, pp. 465-478, 1978.

[13] J. Benesty, J. Chen, and Y. Huang, "A Generalized MVDR Spectrum," IEEE Signal processing letters vol. 12, no. 12, pp 827-830, Dec. 2005.

[14] S. M. Kay, Fundamentals of Statistical Signal Processing Estimation Theory. Prentice-Hall, Engle-wood Cliffs, NJ, 1993.