

# Adding Edges for a Simple Cycle to a Complete Binary Tree Maximizing Total Shortening Path Length

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**Abstract**—This study proposes a model of adding edges of forming a simple cycle to a level of depth  $N$  in a complete binary tree of height  $H$  under giving priority to edges between two nodes of which the deepest common ancestor is deeper. An optimal depth  $N^*$  is obtained by maximizing the total shortening path length which is the sum of shortening lengths of shortest paths between every pair of all nodes in the complete binary tree.

**Index Terms**—complete binary tree, simple cycle, shortest path, organization structure

## I. INTRODUCTION

The pyramid organization structure based on the principle of unity of command can be expressed as a rooted tree, if we let nodes and edges in the rooted tree correspond to members and relations between members in the organization respectively [1, 2]. Then the path between each node in the rooted tree is equivalent to the route of communication of information between each member in the organization. Moreover, adding edges to the rooted tree is equivalent to forming additional relations other than that between each superior and his subordinates.

The purpose of our study is to obtain an optimal set of additional relations to the pyramid organization such that the communication of information between every member in the organization becomes the most efficient. This means that we obtain a set of additional edges to the rooted tree minimizing the sum of lengths of shortest paths between every pair of all nodes.

We have obtained an optimal depth for each of the following three models of adding relations in a level to the organization structure which is a complete  $K$ -ary tree of height  $H$ : (i) a model of adding an edge between two nodes with the same depth, (ii) a model of adding edges between every pair of nodes with the same depth, and (iii) a model of adding edges between every pair of siblings with the same depth [3]. A complete  $K$ -ary tree is a rooted tree in which all leaves have the same depth and all internal nodes have  $K$  ( $K = 2, 3, \dots$ ) children [4]. A complete  $K$ -ary tree of  $K = 2$  is a complete binary tree. Furthermore, we have proposed a model of adding edges between one node and every other node with the same depth in a complete  $K$ -ary tree of height  $H$  [5]. This

model is equivalent to forming additional relations between a representative and the other members in the same level.

This study proposes a model of adding edges of forming a simple cycle to a level of depth  $N$  ( $N = 1, 2, \dots, H$ ) in a complete binary tree of height  $H$  ( $H = 1, 2, \dots$ ) under giving priority to edges between two nodes of which the deepest common ancestor is deeper as follows.

*Step 1:* Add new edges between each pair of nodes of which the depth of the deepest common ancestor is  $N - 1$ .

*Step 2:* Add new edges between each one pair of nodes of which the depth of the deepest common ancestor is  $N - 2$ .

*Step 3:* Add new edges between each one pair of nodes of which the depth of the deepest common ancestor is  $N - 3$  and on which the only added edge is incident.

*Step 4:* Repeat *Step 3* while the depth of the deepest common ancestor is  $N - 4, N - 5, \dots, 1, 0$ .

*Step 5:* Add a new edge between two nodes on which the only added edge is incident.

Fig. 1 shows an example of adding edges of forming a simple cycle to the level of  $N = 3$  in a complete binary tree of  $H = 5$  as *Step 1-5*. In Fig. 1 bold edges between nodes of depth  $N = 3$  signify the added edges. The above model corresponds to the formation of a cycle connection in the same level.

If  $l_{i,j}$  ( $= l_{j,i}$ ) denotes the path length, which is the number of edges in the shortest path from a node  $v_i$  to a node  $v_j$  ( $i, j = 1, 2, \dots, 2^{H+1} - 1$ ) in the complete binary tree

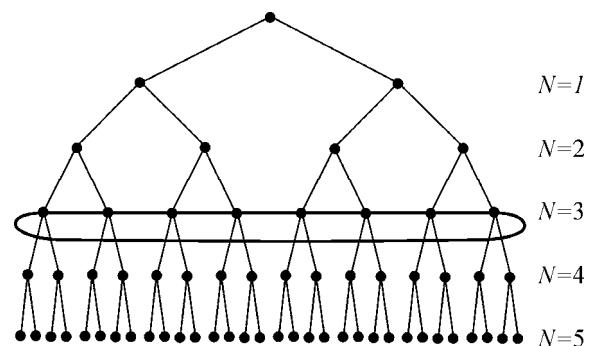


Figure 2. An example of adding edges of forming a simple cycle.

of height  $H$ , then  $\sum_{i < j} l_{ij}$  is the total path length. Furthermore, if  $l'_{ij}$  denotes the path length from  $v_i$  to  $v_j$  after adding edges in this model,  $l_{ij} - l'_{ij}$  is called the shortening path length between  $v_i$  and  $v_j$ , and  $\sum_{i < j} (l_{ij} - l'_{ij})$  is called the *total shortening path length*. Minimizing the total path length is equivalent to maximizing the total shortening path length.

In Section 2 the total shortening path length is formulated when new edges are added to a level of depth  $N$  in a complete binary tree of height  $H$  as *Step 1-5*. In Section 3 an optimal depth  $N^*$  which maximizes the total shortening path length is obtained.

II. FORMULATION OF TOTAL SHORTENING PATH LENGTH

When we add a new edge between two nodes with depth  $N$  of which the depth of the deepest common ancestor is  $N - i$  ( $i = 1, 2, \dots, N$ ), the sum of shortening path lengths can be formulated by summing up the following three equations:

$$A_{H,N}(i) = m(H - N)^2(2i - 1), \tag{1}$$

$$B_{H,N}(i) = 2m(H - N) \sum_{j=1}^{i-1} \{m(H - N + j - 1) + 1\} \times (2i - 2j - 1), \tag{2}$$

and

$$C_{H,N}(i) = \sum_{j=1}^{i-2} \{m(H - N + j - 1) + 1\} \times \sum_{k=1}^{i-j-1} \{m(H - N + k - 1) + 1\} (2i - 2j - 2k - 1), \tag{3}$$

where  $m(h)$  denotes the number of nodes of a complete binary tree of height  $h$  ( $h = 0, 1, 2, \dots$ ), and we define

$$\sum_{j=1}^0 \cdot = 0 \text{ and } \sum_{j=1}^{-1} \cdot = 0.$$

When one edge between two nodes of which the depth of the deepest common ancestor is  $N - i$  in *Step 2-4* is added, the sum of shortening path lengths is obtained by adding  $D_{H,N}(i) + E_{H,N}(i)$  to  $A_{H,N}(i) + B_{H,N}(i) + C_{H,N}(i)$ .  $D_{H,N}(i)$  and  $E_{H,N}(i)$  which are the sum of additional shortening path lengths of using former added edges are given by

$$D_{H,N}(i) = 2m(H - N) \sum_{j=1}^{i-2} m(H - N + j - 1) + P_{H,N}(i), \tag{4}$$

and

$$E_{H,N}(i) = \begin{cases} 0 & (i = 1) \\ 3m(H - N)^2 & (i = 2) \\ 7m(H - N)^2 + Q_{H,N} & (i = 3) \\ 9m(H - N)^2 + R_{H,N}(i) & (i = 4, 5) \\ 7m(H - N)^2 + R_{H,N}(i) & (i = 6) \\ 6m(H - N)^2 + R_{H,N}(i) & (i \geq 7) \end{cases}, \tag{5}$$

where

$$P_{H,N}(i) = \begin{cases} 0 & (i \leq 3) \\ m(H - N)^2(i - 3) & (i \geq 4) \end{cases}, \tag{6}$$

$$Q_{H,N} = 4m(H - N)m(H - N + 1), \tag{7}$$

and

$$R_{H,N}(i) = m(H - N)\{4m(H - N + i - 2) + 4m(H - N + i - 3) + 2m(H - N + i - 4)\}, \tag{8}$$

and we define  $\sum_{j=1}^0 \cdot = 0$  and  $\sum_{j=1}^{-1} \cdot = 0$ . Therefore the sum of shortening path lengths of *Step 1-4* is given by

$$F_{H,N} = \sum_{i=1}^N 2^{N-i} \{A_{H,N}(i) + B_{H,N}(i) + C_{H,N}(i) + D_{H,N}(i) + E_{H,N}(i)\}. \tag{9}$$

The sum of shortening path lengths of *Step 5* is given by

$$G_{H,N} = A_{H,N}(N) + B_{H,N}(N) + C_{H,N}(N) + D_{H,N}(N) + E_{H,N}(N) - L_{H,N}, \tag{10}$$

where  $L_{H,N}$  is the sum of shortening path lengths of subtracting repeated adding shortening path lengths and is shown as

$$L_{H,N} = \begin{cases} m(H - N)^2 & (N = 1) \\ 6m(H - N)^2 & (N = 2) \\ 14m(H - N)^2 & (N = 3) \\ 2m(H - N)^2 & (N \geq 4) \end{cases}. \tag{11}$$

From the above equations, the total shortening path length of this model  $S_{H,N}$  is formulated by

$$S_{H,N} = F_{H,N} + G_{H,N}. \tag{12}$$

III. AN OPTIMAL ADDING DEPTH

Since the number of nodes of a complete binary tree of height  $h$  is

$$m(h) = 2^{h+1} - 1, \tag{13}$$

$S_{H,N}$  becomes

$$S_{H,N} = \alpha_{H,N} + \beta_{H,N} + \gamma_{H,N} + \theta_{H,N}, \quad (14)$$

where

$$\begin{aligned} \alpha_{H,N} &= \sum_{i=1}^N 2^{N-i} \{A_{H,N}(i) + B_{H,N}(i) + C_{H,N}(i)\} \\ &+ A_{H,N}(N) + B_{H,N}(N) + C_{H,N}(N) \quad (15) \\ &= (3N^2 + 7N - 2)2^{2H-N-1} - (3N - 1)2^{H+1} \\ &+ 3 \cdot 2^N - 4, \end{aligned}$$

$$\beta_{H,N} = \begin{cases} 0 & (N \leq 2) \\ \sum_{i=3}^N 2^{N-i} \{D_{H,N}(i) + Q_{H,N}\} + D_{H,N}(N) \\ = -2^{2H-2N+3} + (2N+1)2^{2H-N} & (N \geq 3) \\ + 2^{H-N+4} - (N+4)2^H + 7 \cdot 2^{N-2} - 6 \end{cases}, \quad (16)$$

$$\gamma_{H,N} = \begin{cases} 0 & (N \leq 3) \\ \sum_{i=4}^N 2^{N-i} R_{H,N}(i) + R_{H,N}(N) \\ = (13N - 26)2^{2H-N-1} & (N \geq 4) \\ - (13N - 16)2^{H-2} + 5 \cdot 2^{N-2} \end{cases}, \quad (17)$$

and

$$\theta_{H,N} = \begin{cases} -2^{2H} + 2^{H+1} - 1 & (N = 1) \\ 0 & (N = 2) \\ 7 \cdot 2^{2H-3} - 3 \cdot 2^{H+1} + 10 & (N = 3) \\ 21 \cdot 2^{2H-5} - 21 \cdot 2^{H-1} + 42 & (N = 4) \\ 43 \cdot 2^{2H-7} - 43 \cdot 2^{H-2} + 86 & (N = 5) \\ 85 \cdot 2^{2H-9} - 85 \cdot 2^{H-3} + 170 & (N = 6) \\ -2^{2H-2N+3} + 171 \cdot 2^{2H-N-4} + 2^{H-N+3} & (N \geq 7) \\ -171 \cdot 2^{H-4} + 171 \cdot 2^{N-6} - 2 \end{cases} \quad (18)$$

Let

$$\Delta S_{H,N} \equiv S_{H,N+1} - S_{H,N}, \quad (19)$$

for  $N = 1, 2, \dots, H - 1$ , so that we have the following results of (I)-(VII).

(I) When  $N = 1$ , then we have

$$\begin{aligned} \Delta S_{H,1} &= \alpha_{H,2} - \alpha_{H,1} - \theta_{H,1} \\ &= 2^{2H+1} - 2^{H+3} + 7 \\ &> 0. \end{aligned} \quad (20)$$

(II) When  $N = 2$ , then we have the following. If  $H = 3$ , then

$$\begin{aligned} \Delta S_{H,2} &= \alpha_{H,3} - \alpha_{H,2} + \beta_{H,3} + \theta_{H,3} \\ &= 3 \cdot 2^{2H-1} - 17 \cdot 2^H + 30 \\ &< 0, \end{aligned} \quad (21)$$

and if  $H \geq 4$ , then

$$\Delta S_{H,2} > 0. \quad (22)$$

(III) When  $N = 3$ , then we have

$$\begin{aligned} \Delta S_{H,3} &= \alpha_{H,4} - \alpha_{H,3} + \beta_{H,4} - \beta_{H,3} + \gamma_{H,4} + \theta_{H,4} - \theta_{H,3} \\ &= -3 \cdot 2^{2H-4} - 43 \cdot 2^{H-1} + 90 \\ &< 0. \end{aligned} \quad (23)$$

(IV) When  $N = 4$ , then we have

$$\begin{aligned} \Delta S_{H,4} &= \alpha_{H,5} - \alpha_{H,4} + \beta_{H,5} - \beta_{H,4} + \gamma_{H,5} - \gamma_{H,4} + \theta_{H,5} \\ &- \theta_{H,4} \\ &= -43 \cdot 2^{2H-5} - 11 \cdot 2^H + 140 \\ &< 0. \end{aligned} \quad (24)$$

(V) When  $N = 5$ , then we have

$$\begin{aligned} \Delta S_{H,5} &= \alpha_{H,6} - \alpha_{H,5} + \beta_{H,6} - \beta_{H,5} + \gamma_{H,6} - \gamma_{H,5} + \theta_{H,6} \\ &- \theta_{H,5} \\ &= -133 \cdot 2^{2H-7} - 83 \cdot 2^{H-3} + 276 \\ &< 0. \end{aligned} \quad (25)$$

(VI) When  $N = 6$ , then we have

$$\begin{aligned} \Delta S_{H,6} &= \alpha_{H,7} - \alpha_{H,6} + \beta_{H,7} - \beta_{H,6} + \gamma_{H,7} - \gamma_{H,6} \\ &+ \theta_{H,7} - \theta_{H,6} \\ &= -1471 \cdot 2^{2H-11} - 83 \cdot 2^{H-3} + 554 \\ &< 0. \end{aligned} \quad (26)$$

(VII) When  $N \geq 7$ , then we have

$$\begin{aligned} \Delta S_{H,N} &= \alpha_{H,N+1} - \alpha_{H,N} + \beta_{H,N+1} - \beta_{H,N} + \gamma_{H,N+1} \\ &- \gamma_{H,N} + \theta_{H,N+1} - \theta_{H,N} \\ &= 3 \cdot 2^{2H-2N+2} - (24N^2 + 144N - 253)2^{2H-N-5} \\ &- 3 \cdot 2^{H-N+2} - 41 \cdot 2^{H-2} + 555 \cdot 2^{N-6} \\ &< 0. \end{aligned} \quad (27)$$

From the above results, the optimal adding depth  $N^*$  which maximizes  $S_{H,N}$  can be obtained and is given in Theorem 1.

*Theorem 1*

- (i) If  $H = 1$ , then the optimal adding depth is  $N^* = 1$ .
- (ii) If  $2 \leq H \leq 3$ , then  $N^* = 2$ .
- (iii) If  $H \geq 4$ , then  $N^* = 3$ .

*Proof:*

- (i) If  $H = 1$ , then  $N^* = 1$  trivially.

TABLE I.  
OPTIMAL ADDING DEPTH  $N^*$

$H$	$N^*$	$S_{H,N^*}$
1	1	1
2	2	16
3	2	120
4	3	758
5	3	3782
6	3	16742
7	3	70310
8	3	288038
9	3	1165862
10	3	4690982
11	3	18819110
12	3	75386918
13	3	301768742
14	3	1207517222
15	3	4830953510
16	3	19325583398
17	3	77305872422
18	3	309230567462
19	3	1236936425510
20	3	4947774013478

(ii) If  $H = 2$ , then  $N^* = 2$  since  $\Delta S_{2,1} > 0$ . If  $H = 3$ , then  $N^* = 2$  since  $\Delta S_{3,1} > 0$  and  $\Delta S_{3,2} < 0$ .

(iii) If  $H \geq 4$ , then  $N^* = 3$  since  $\Delta S_{H,N} > 0$  for  $N \leq 2$  and  $\Delta S_{H,N} < 0$  for  $N \geq 3$ .

Q.E.D.

Table 1 shows the optimal adding depth  $N^*$  and the total shortening path lengths  $S_{H,N^*}$  in the case of  $H=1, 2, \dots, 20$ .

#### IV. CONCLUSIONS

This study considered the addition of relations to a pyramid organization structure such that the communication of information between every member in the organization becomes the most efficient. For a model of adding edges of forming a simple cycle to a level of depth  $N$  in a complete binary tree of height  $H$  under giving priority to edges between two nodes of which the deepest common ancestor is deeper, we obtained an optimal depth  $N^*$  which maximizes the total shortening path length in Theorem 1. This result indicates the most efficient way of adding relations of forming a simple cycle in a level is to use the first level, the second level, or the third level depending on the number of levels in the organization structure.

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