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Intuition – Based Teaching Mathematics for Engineers

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Abstract - **It is suggested to teach Mathematics for engineers based on development of mathematical intuition, thus, combining conceptual and operational approaches. It is proposed to teach main mathematical concepts based on discussion of carefully selected case studies following solving of algorithmically generated problems to help mastering appropriate mathematical tools. The former component helps development of mathematical intuition; the latter applies means of adaptive instructional technology to improvement of operational skills. Proposed approach is applied to teaching uniform convergence and to knowledge generation using Computer Science object-oriented methodology.**

I. Introduction

Successful teaching and learning mathematics comprises two indistinguishable components: connecting mathematics to the real world and mastering pure mathematics. Each of them requires not only time and efforts, but also specific learning abilities to grasp variety of notions and concepts from different fields of Mathematics. For engineering students, finding an efficient way of teaching and learning Mathematics is of crucial importance. On one hand, they need Mathematics to master their major disciplines; on the other hand, they do not have sufficient intention and time to study Mathematics rigorously. With these contradictory conditions in mind, it is proposed in this paper to make a stress on intuitive perception of Mathematics combined with computer-supported mastering of its essential applications tools.

Theory and practice of mathematical intuition is subject of intensive research [1-18] that can be traced back to 1852. In this paper, we define mathematical intuition "as ability to sense or know immediately without reasoning", [19]. More specifically, we treat mathematical intuition as ability to solve correctly mathematical problems or applications without rigorous reasoning. Publication [20] provides a typical example. Its author, a mathematician from the Princeton University developed mathematical theory of Nim game underlain by the binary number system. In the footnotes he mentions: ''The modification of the game . . . was described to the writer by Mr. Paul E. More in October, 1899. Mr. More at the same time gave a method of play which, although expressed in a different form, is really the same as one used here, but he could give no proof of this rule'', [20, p. 35]. Thus, the author of the well-known mathematical paper has not actually solved the problem; he just proved ***the*** validity of the proposed solution. The solution itself was obtained by mathematical intuition of the other person. Publication [21] provides more details.

Methodologically, suggested approach is based on Constructivism [22 - 25], the Theory of Knowledge Spaces [26 - 29], and psychology of Problem Solving [30]. Pedagogically, it relies on detailed exploration of carefully selected case studies. Examples furnished below serve presentation of the proposed approach and are related to uniform convergence of Rieman sums and using Computer Science object-oriented methodology as an educational tool of Mathematics.

The paper is organized as detailed description of the proposed examples, one per section. Publication [30] related to adaptive educational system ALEKS estimates the number of case studies for Precalculus course as 265. This number can serve as a rough estimate for other courses as well.

II. Uniform convergence in teaching integration

Integration is one of the main topics of Calculus and is an important component of mathematical preparation of engineering students. In Calculus, in contrary to Mathematical Analysis courses, main notions and concepts of the theory of integration are not presented rigorously. To some extent, such approach can be justified: Calculus students should only know essentials of the theory and be able to apply them to applications. But oversimplification plays a negative role: students that studied Calculus in this way are unable to apply it creatively enough in engineering.

Calculus distinguishes from other courses, like Algebra or Precalculus, in using limit as a basic concept. From Algebra course, students get elementary understanding of a limit related to infinite geometric sequences: If in a sequence $S_n = a + aq + aq^2 + \ldots + aq^n$ the number of terms n increases to infinity and $|q| < 1$, then the magnitude of S_n approaches the value of $S = a/(1-q)$. Students are comfortable with limit-notation $\lim S_n = S$ with $n \to \infty$, and this construct works well while students remain in the boundaries of theory of limits or differentiation. Extension of the previous concept to a limit of a function of one variable does not usually cause comprehension problems. Students perceive easily that $\lim f(x)$ with $x \to a$ is an *expected* value of $f(x)$ when x approaches a. Further extension of the concept to the case of difference quotient in the definition of the derivative function is accepted adequately also.

A situation becomes quite different when Integral Calculus begins, and the definite integral is introduced. Studying the definite integral, students find themselves in a quite different environment. The definite integral is usually introduced in Calculus using Riemann sums. The Riemann sum includes partition of an interval in the domain of a function into small subintervals, selection of a set of points, one point from each subinterval, and summation of the products: width of a subinterval times the value of a function calculated in the selected point. The definite integral is a limit of the corresponding Riemann sum with the widths of all subintervals approaching zero. In this situation, students actually need to grasp the following definition: The definite integral of a function *f(x)* from *a* to *b* is a limit of the Riemann sum

$$\lim_{\max|\Delta x_i|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i \ . \qquad (1)$$

For many Calculus students there is a pitfall. The notion of the limit that has been just used in the previous Calculus sections is not the same as that in the definite integral. In particular, The Riemann sum comprises more elements than a function *f(x)*. It includes the function *f(x)*, a partition $\{\Delta x_i\}$, a selection $\{x_i^*\}$, and an natural number *n*, the number of elements in the partition. Thus, the Riemann sum is *not a function* of the parameter $max|\Delta x_i|$ that approaches 0 when the limit is taken. Also, it is not self-evident what the interplay among partition, selection, and the sum of products in the Riemann sum is. It is not fully clear to students how each of them affects the existence and the value of the limit.

Some Calculus textbooks try to simplify the situation by imposing restrictions on partition and selection aimed at making it look like previous cases. The most popular way is to allow for only equal-sized elements $\Delta x_i = \Delta x$ of the partition: $\Delta x_i = \Delta x=(b-a)/n$. In this case, the Riemann sum looks like a function of *n*:

$$R_n = \lim_{\max|\Delta x_i|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{\Delta x\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty} \frac{(b-a)}{n} \sum_{i=1}^{n} f(x_i^*) \ . \qquad (2)$$

But this is not really so, because arbitrary selection $\{x_i^*\}$ remains, and does not allow the Riemann sum R_n to be a function of *n*. Some simplifications impose restrictions on selections $\{x_i^*\}$ as well. For instance, they require that points x_i^* were left or right endpoints of the corresponding subintervals Δx_i. The other approach is to define the definite integral for continuous functions only. For example, a popular textbook [31] defines the definite integral for continuous functions only using equal-sized subintervals for partition. This definition is misleading because the definite integral exists for discontinuous functions as well. Moreover, additional explanation regarding the choice of subintervals makes things worse, rather than help understanding the notion. It reads: "Although we have <used> subintervals of equal width, there are situations in which it is advantageous to work with subintervals of unequal width. For instance,…NASA provided velocity data at times that were not equally spaced. … And there are methods for numerical integration that take advantage of unequal subintervals", [31, p.328]. A thoughtful student will probably be confused with such explanation. A question may be this: Is it NASA (The National Aeronautic and Space Administration of the United States) that sets up the *mathematical definition* of the definite integral? The other popular restriction is the use of "left-hand" or "right-hand" Riemann sums. In a left-hand Riemann sum, $t_i = x_i$ for all *i*, and in a right-hand Riemann sum, $t_i = x_{i+1}$ for all *i*. Publication [32] summarizes simplified approaches to the definition of the definite integral as follows. "Some calculus books … limit themselves to specific types of tagged partitions. If the type of partition is limited too much, some non-integrable functions may appear to be integrable."

The main problem stems from the fact that the Riemann sum is neither a *sequence* nor a *function of the parameter* that approaches zero. The limit used in the definition of the definite integral is a limit of a new type, not known to Calculus students. It is a *uniform limit* resulting from *uniform convergence* of the Riemann sums with regards to partition and selection. This new type of limit requires special consideration and additional explanations aimed to prepare students for adequate perception of the definite integral. Early and informal introduction of uniform convergence of the Riemann sums is based on a series of carefully selected examples and exercises, and gradually develops student mathematical intuition related to uniform convergence. It uses the notion of infinitesimals and their summation, [33]: "…an *infinitesimal quantity* is one which, while not necessarily coinciding with zero, is in some sense smaller than any finite quantity. … An infinitesimal is a quantity so small that its square and all higher powers can be neglected." The following example demonstrates a possible way of classroom discussion.

Example. An automatic device measures a distance covered by an airplane. To do that, it measures a speed of the airplane based on air pressure in consecutive short time intervals, multiplies the speed by the length of the time interval, and adds up the products to calculate a distance. Example of calculations is given in table 1 for time for total time of 1 hour and time intervals of 10 min. The infinitesimals $D_i=V_i\times\Delta t_i$ are the products of speed by time interval, respectively. There are six time subintervals in the table, so that the estimated distance based on 10 minutes time intervals is

$$S_6 = \sum_{i=1}^{6} D_i = \sum_{i=1}^{6} V_i \times \Delta t_i = 546.667 \text{ miles}, \qquad (3)$$

where V_i stands for a speed measured at a randomly selected moment time t_i^* inside the time interval $\Delta t_{i,}$. To obtain more precise result, we should measure speed more and more frequently, say each 1 min, then each 30 sec etc. By doing so, we may get results presented in table 2 labeled as S60, S120, S360, and S3600, respectively. Although we cannot provide an analytical formula for calculation of the limit value of S_n, we can guess, based on table 2, that it is close to 550 miles:

TABLE 1
CALCULATION OF A DISTANCE

Measure-ment	Time interval, min	Speed, mph	Distance, miles
i	Δt_i	V_i	$D_i=V_i\times\Delta t_i$
1	10	500	10/60×500=83.333
2	10	540	10/60×540=90.000
3	10	560	10/60×560=93.333
4	10	490	10/60×490=81.667
5	10	610	10/60×610=101.667
6	10	580	10/60×580=96.667
Total	60		546.667

Total time = 1 h, time interval = 10 min

TABLE 2

SPEED MEASUREMENTS WITH DIFFERENT TIME INTERVALS

Time interval	Number of measurements	Distance estimation, miles
Δti	n	S_n
10 min	6	546.667
1 min	60	552.345
30 sec	120	549.678
10 sec	360	550.204
1 sec	3600	549.943

Total time = 1 h.

This means that by making speed measurements infinitely frequently, we make each time interval Δt_i, and each partial distance D_i infinitesimally small while increasing the number of time intervals to infinity. By doing so, we approach the exact value of the distance covered in one hour that is close to 550 miles.

It is important to stress at this point that though speed can be measured at an *arbitrary moment of time* inside each time interval, it will not affect the final result. With time intervals becoming smaller and smaller, random choice of a moment of speed measurement will not have any impact on the total distance. The reason is this: within a very small time interval the speed of an airplane remains about the same. The time intervals *need not be necessarily equal*; the only condition is that all of them, or, in other words, maximal subinterval, should vanish.

Equipped with the knowledge of uniform convergence, students are prepared to perceive the following definition of the definite integral as a uniform limit of the Riemann sum:

$$\int_a^b f(x)dx = \lim_{\max|\Delta x_i|\to 0} \sum_{i=0}^{N} f(x_i^*)\Delta x_i . \qquad (5)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} D_i = \lim_{n\to\infty} \sum_{i=1}^{n} V_i \times \Delta t_i \approx 550 \qquad (4)$$

Uniformness means that the limit does not depend on either choice of subintervals Δx_i or points x_i^* inside them. For Calculus needs it is sufficient just to mention that the uniform limit exists for continuous functions or for functions with no more than countable amount of jumps. Integrable functions are necessary bounded. The necessary and sufficient conditions, the Lebesgue's Criterion, may be found in [34].

III. OBJECT-ORIENTED APPROACH IN TEACHING MATHEMATICS

In this section we present an approach to teaching mathematics that may be attractive for engineering students pursuing careers in computer programming, information systems, or information technology. It is based on similarities between object-oriented programming and mathematical systems. Object-oriented programming (OOP) operates with "*classes*" and "*objects*" that possess "*properties*" and are subject to operations called "*methods*". In this section, a "method" means a set of computer statements aimed at performing a certain task. A class defines general characteristics of a problem in question, while object represents a particular instance of a class. We will show below that a particular area of Mathematics follows same rules that the OOP does. This area of Mathematics studies abstract mathematical systems. The last are triples containing a set of elements, a set of axioms related to the elements, and a set of operations over the elements of the system. Mapping of OOP class on a mathematical system is this. Objects of a class are considered as a set of elements of a mathematical system; properties, as its axioms, and methods, as applicable operations. With these similarities in mind, OOP may serve as a "tangible" tool for presentation of abstract mathematical systems. Suggested approach is useful for engineers in view of development of computer – based proof in pure mathematics, as discussed in [35].

In this section below, a problem borrowed from [36] is used as an example. The problem is this. "Three friends, Jane, Rose and Phyllis, study different languages and have different career goals. One wants to be an artist, one a doctor, and the third a lawyer. <The following rules determine their choices:> (1) The girl who studies Italian does not plan to be a lawyer; (2) Jane studies French and does not plan to be an artist; (3) The girl who studies Spanish plans to be a doctor; (4) Phyllis does not study Italian. Find the language and career goal of each girl."

To apply object-oriented approach, we first introduce a class **Student** with two properties: ***Language*** and ***Career***. Property-1 ***Language*** has values {*French, Italian, Spanish*}, property-2 ***Career***, values {*Artist, Doctor, Lawyer*}. The class has three instances, or objects, in this problem that may be labeled by the first letters of the girls' names: **J**, **R**, and **P**, correspondingly. A method called **MatchGoals()** sets up correspondence among the properties of the objects. It works with a

3 by 3 matrix with rows corresponding to the values of property-1 (***Language***), and columns corresponding to property-2 (***Career***). An algorithm of the method processes the rules successfully and fills in the elements of the matrix with letters **J**, **P,** or **R** corresponding to the objects. Eventually, only three elements of the matrix are filled in, each in one row and one column; all other elements will be set empty. The algorithm performs several runs using the rules consequently and analyzes the matrix after each run, as shown in Table 3, where we use logical symbols ~(NOT), $\wedge$ (AND) and $\vee$ (OR), while symbol "-" stands for "**Empty**". The last means that the element of the table contains no object name and should not be analyzed in consecutive runs.

The solution process starts with Run 1 of the method **MatchGoals()** that uses Rule (1) and empties an element located at the intersection of row *Italian* and column *Lawyer*. Then Run 2 uses Rule (2) and fills in row *French* with **J** and column *Artist* with **~J**. After that the method performs analysis of the table and changes the element *French-Artist* for **Empty**, because it contains a contradictory condition **J** $\wedge$ **~J**.

Run 3 utilizes Rule (3), and labels the element *Spanish-Doctor* as a candidate, though not assigning it any value. Though a value of this element is not known at this stage, labeling it as a chosen element allows marking all the rest elements in row *Spanish* and column *Doctor* as **Empty**, see Table 3, Analysis 2a. Now, the choice of non-empty elements of the matrix is complete because each row and column contains only one non-**Empty** element. The next step of the analysis fill in the element *French-Lawyer* is **J**; see Table 3, Analysis 2b.

Run 4 fills in an element *Italian-Artist* with **~P**, using the Rule (4) making it equal to **~J** $\wedge$ **~P**. This allows for its replacement with **R** in Analysis 3 step; see Table 3, Analysis 3a. The last finalizes the process by matching remaining object **P** with the element *Spanish-Doctor*, see Table 3, Analysis 3b. The problem is solved; the solution has been obtained completely algorithmically though the problem looked like a puzzle in the beginning.

For the objectives of this paper, polymorphism of OOP classes is important. The last means that class can be designed independently from its instantiations. In our case, object-oriented approach allows for easy generalization of a particular problem to numerous applications. Thus, it may be reconsidered as a class **WildLife** that inherits all properties and methods of the previous class **Student**. Property-1 may be renamed as ***NumberOfLegs*** with a set of values {*0,2,4*}, while property-2, as ***TypeOfMoving*** with values {*Flying, Walking, Swimming*}. The objects of the new class may be named as **Animal**, **Bird**, and **Fish**. The new problem is assigning a number of legs and type of moving to each creature. It can be solved using exactly the same *Run – Analysis* sequence as above. It results in assigning 4 legs and W*alking* as type of moving to **Animal**, 2 and *Flying* to **Bird**, and 0 and *Swimming* to **Fish**, respectively.

We can continue with other examples of the same class with different objects and different interpretations of its properties. Rules can be changes as well, but it should be stressed there should nether be too little rules, nor contradictory ones. The number of rules needs to be sufficient to fill in all elements of the matrix. On the other hand, too many rules may be redundant or contradictory. Different collections of rules may be equivalent, leading to the same final result. As a research project, students may be asked to change the subject area, the type of the participants, and the properties so that a new problem had the same or similar sequence of *Run – Analysis* steps needed for its solution. Engineering students learning Mathematics in such way will be able to recognize similarity of mathematical models and use similar logical reasoning in different situations independently from subject areas, like Biology, Chemistry, Information Technology, Physics etc.

TABLE 3
RUNS OF THE **MatchGoals()** METHOD

Run 1. Using Rule (1).

	Artist	*Doctor*	*Lawyer*
French			
Italian			-
Spanish			

Run 2. Using Rule (2).

	Artist	*Doctor*	*Lawyer*
French	**J** $\wedge$ **~J**	**J**	**J**
Italian	**~J**		-
Spanish	**~J**		

Analysis 1. Logical analysis and transformations.

	Artist	*Doctor*	*Lawyer*
French	-	**J**	**J**
Italian	**~J**		-
Spanish	**~J**		

Run 3. Using Rule (3).

	Artist	*Doctor*	*Lawyer*
French	-	**J**	**J**
Italian	**~J**		-
Spanish	**~J**		

Analysis 2a. Logical analysis and transformations

	Artist	*Doctor*	*Lawyer*
French	-	-	**J**
Italian	**~J**	-	-
Spanish	-		-

Analysis 2b. Logical analysis and transformations.

	Artist	*Doctor*	*Lawyer*
French	-	-	**J**
Italian	**~J**	-	-
Spanish	-		-

Run 4. Using Rule (4).

	Artist	*Doctor*	*Lawyer*
French	-	-	**J**
Italian	**~J** $\wedge$ **~P**	-	-
Spanish	-		-

Analysis 3a. Logical analysis and transformations (continued.)

	Artist	*Doctor*	*Lawyer*
French	-	-	**J**

Italian	**R**	-	-
Spanish	-		-

Analysis 3b. Logical analysis and transformations Final result.

	Artist	*Doctor*	*Lawyer*
French	-	-	**J**
Italian	**R**	-	-
Spanish	-	**P**	-

IV. CONCLUSIONS

It is suggested to teach mathematics for engineers based on development of mathematical intuition using adaptive educational computer systems as a tool of mastering operational skills. Suggested approach requires development of collections of case studies each covering a specific course of Mathematics. Literature sources allow for assumption that 200 – 300 case studies are sufficient to cover a typical course of undergraduate mathematics. Combining intuition-based teaching with computerized educational systems allows for better mathematics preparation of engineers and makes teaching and learning process attractive, student-paced, and textbook independent.

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